The University Rankings Game:
Modeling the Competition among Universities for Ranking

APPENDIX
Adjacent Category Logit Model

Let:

- u = 1… U as the total number of universities that were ranked for every year in the observation period.
- t = 1… T as the number of the time period for which we observe the university ranks (T = 8 in our sample).
- r = 1… R as indexing ranks, where for top 50 universities we study R = 50.
- X_{ut} = the matrix of explanatory variables for university u at time t. (These variables are the ones used by USNews)
- π_r = the probability of observing rank r, such that $\pi_r \geq 0$, $\forall$ r = 1… R and $\sum_{r=1}^{R} \pi_r = 1$.

For adjacent categories r and $r+1$, we define the adjacent-categories log-odds unit (LOGIT) as (Goodman 1983; Simon 1974):

$$\text{(1)} \qquad \operatorname{logit}\left[\frac{P(Y_{ut} = r \mid X_{ut}, Y_{u(t-1)})}{P(Y_{ut} = (r+1) \mid X_{ut}, Y_{u(t-1)})}\right] = \log\left(\frac{\pi_r(X_{ut}, Y_{u(t-1)})}{\pi_{r+1}(X_{ut}, Y_{u(t-1)})}\right),$$

where, Y_{ut} is the rank for the university u at a given year t and

$\pi_r(X_{ut}, Y_{u(t-1)}) = P(Y_{ut} = r \mid X_{ut}, Y_{u(t-1)})$.

To incorporate explanatory variables, the logarithm is specified as a linear function of the explanatory variables (e.g., McCullagh 1980):

$$\text{(2)} \qquad \log\left(\frac{\pi_r(X_{ut}, Y_{u(t-1)})}{\pi_R(X_{ut}, Y_{u(t-1)})}\right) = \alpha_r + Y_{u(t-1)}\beta_r + X_{ut}\gamma_r + Y_{u(t-1)}X_{ut}\delta_r.$$

As is the case in logit models with multiple categories, $\pi_r(X_{ut}, Y_{u(t-1)})$ is defined as:

$$\text{(3)} \qquad \pi_r(X_{ut}, Y_{u(t-1)}) = P(Y_{ut} = r \mid X_{ut}, Y_{u(t-1)}) = \frac{\exp(\alpha_r + Y_{u(t-1)}\beta_r + X_{ut}\gamma_r + Y_{u(t-1)}X_{ut}\delta_r)}{1 + \sum_{r=1}^{R-1}\exp(\alpha_r + Y_{u(t-1)}\beta_r + X_{ut}\gamma_r + Y_{u(t-1)}X_{ut}\delta_r)}.$$

The probability specification in Equation 4 leads to the familiar likelihood function (L):

(4) $$L = \prod_{u=1}^{U}\prod_{t=2}^{T}\prod_{r=1}^{R}[\pi_r(X_{ut}, Y_{u(t-1)})]^{I_{utr}},$$

where, I_{utr} is an indicator function that equals 1 if $Y_{ut} = r$, else it equals 0.

As we model lag of rank as an explanatory variable in Equation 4, $t = 2,...T$. To recognize the hierarchy inherent in the ranking we specify $\beta_r = (R-r)\beta$ and $\gamma_r = (R-r)\gamma$. Once parameter estimates are available by maximizing the logarithm of Equation 4, we calculate the probability of change in rank as:

(5) $$\frac{\pi_r(X_{ut}, Y_{u(t-1)})}{\pi_{r+p}(X_{ut}, Y_{u(t-1)})} = \exp(\alpha_r + pY_{u(t-1)}\beta + pX_{ut}\gamma + pY_{u(t-1)}X_{ut}\delta).$$

With $Y_{u(t-1)} = r$ modeled as an explanatory variable, we can use Equation 8 to calculate the probability of $Y_{ut} = r + p$, $\forall$ *p = 0, 1, 2…,* $\forall$*r = 1,…R-1.*